On an existence theorem of global strong solution to the magnetohydrodynamic system in three dimensional exterior domain

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Abstract

In this paper we study the initial-boundary value problem for the magnetohydrodynamic system in three dimensional exterior domain. We show an existence theorem of global in time strong solution for small L^3-initial data and we also show its asymptotic behavior when time goes to infinity.

Keywords and phrases: magnetohydrodynamic system, L^q-L^r estimates, analytic semigroup, global existence, exterior domain, viscous incompressible and electrically conducting fluids.
2000 Mathematics Subject Classification: 35Q30,76D03,76W05

1 Introduction and main results

Let $\mathcal{O}$ be a *simply connected* and bounded open set in $\mathbb{R}^3$ with $C^{2,1}$-boundary. We choose some $R_0 > 0$ such that $\mathcal{O} \subset B_{R_0} = \{x \in \mathbb{R}^3 \mid |x| < R_0\}$ and fix it. Let Ω be the exterior domain to $\mathcal{O}$, i.e., $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{O}}$. In this paper we are concerned with the initial-boundary value problem of the magnetohydrodynamic system (the Ohm-Navier-Stokes system) concerning the velocity $\boldsymbol{v} = (v_1(x,t), v_2(x,t), v_3(x,t))$, pressure $p = p(x,t)$ and magnetic field $\boldsymbol{B} = (B_1(x,t), B_2(x,t), B_3(x,t))$ in $\Omega \times (0,\infty)$:

$$\begin{cases} \boldsymbol{v}_t - \Delta \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p + \boldsymbol{B}\times \operatorname{curl}\boldsymbol{B} = 0 & \text{in} \quad \Omega\times(0,\infty), \\ \boldsymbol{B}_t + \operatorname{curl}\operatorname{curl}\boldsymbol{B} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{v} = 0 & \text{in} \quad \Omega\times(0,\infty), \\ \operatorname{div}\boldsymbol{v} = 0, \quad \operatorname{div}\boldsymbol{B} = 0 & \text{in} \quad \Omega\times(0,\infty), \\ \boldsymbol{v} = 0, \quad \boldsymbol{\nu}\cdot\boldsymbol{B} = 0, \quad \operatorname{curl}\boldsymbol{B}\times\boldsymbol{\nu} = 0, & \text{on} \quad \partial\Omega\times(0,\infty), \\ \boldsymbol{v}(x,0) = \boldsymbol{a}, \quad \boldsymbol{B}(x,0) = \boldsymbol{b} & \text{in} \quad \Omega. \end{cases} \tag{MHD}$$

Here $\boldsymbol{a} = (a_1(x), a_2(x), a_3(x))$ and $\boldsymbol{b} = (b_1(x)), b_2(x), b_3(x))$ are the prescribed initial data for the velocity and magnetic field, respectively and $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ is the unit outer normal on $\partial\Omega$. The magnetohydrodynamic system is known to be one of the mathematical models describing the motion of the incompressible viscous and electrically conducting Newtonian fluids. This system is a coupled system of the Navier-Stokes system, Maxwell's equations and Ohm's law under the MHD approximation (see e.g., Landau and Lifshitz [14]).

On the nonstationary problem of the magnetohydrodynamic system, there are many works when $\Omega = \mathbb{R}^3$ or Ω is bounded. For example, Ladyzhenskaya and Solonnikov [13], Duvaut and J.-L. Lions [4] and Sermange and Temam [19]. However, all of the works above are done in the L^2 setting. While on the other hand, Yoshida and Giga [24] studied (MHD) when Ω is bounded by analytic semigroup approach similar to Giga and Miyakawa [7] and they constructed the unique global strong solution if the initial data $(\boldsymbol{a}, \boldsymbol{b})$ are sufficiently small in sense of L^3. In the exterior domain case, Kozono [12] showed the energy decay of the weak solution of (MHD). As far as the author knows, there has been no work on a global in time existence of strong solution to (MHD) when Ω is exterior domain.

For the nonstationary problem of the Navier-Stokes equations for the motion of the viscous incompressible fluids, T. Kato [11] showed the global solvability of the Cauchy problem if initial velocity $\boldsymbol{a}$ is sufficiently small with respect to L^n-norm ($n \geq 2$ denotes the dimension). The argument of Kato is based on the estimates of various L^q-norm of the Stokes semigroup (in the whole space, the Stokes semigroup is essentially the same as the heat semigroup $e^{t\Delta}$). In particular, the L^q-L^r type estimates for such semigroup play a crucial role in his argument. The result of Kato was extended to the case of n-dimensional exterior domain ($n \geq 3$) by Iwashita [10]. Iwashita showed the L^q-L^r estimates for the Stokes semigroup in exterior domain which will be introduced later and solved the initial boundary value problem of the Navier-Stokes equations in exterior domain by using Kato's iteration scheme. In view of Kato and Iwashita, if the initial value $(\boldsymbol{a}, \boldsymbol{b})$ are small enough in the sense of the L^3-norm, we can expect that (MHD) admits a unique global strong solution. Indeed, as mentioned before Yoshida and Giga [24] succeeded in constructing the global L^3-solution when Ω is bounded domain. Thus, our main purpose of the present paper is to show an existence theorem of global strong solution for (MHD).

Since the main point of the argument of Kato and Iwashita consists of the study of the linearized problem. Therefore in order to treat (MHD) by such argument, we have to study the linearized problems of (MHD) and investigate the properties of solutions to such problem. If we linearize (MHD), we obtain two systems of equations. The first one is system of the Stokes equations and the second one is the following linear

diffusion equations with the perfectly conducting wall:

$$\begin{cases} \boldsymbol{u}_t + \operatorname{curl}\operatorname{curl}\boldsymbol{u} = 0, \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \boldsymbol{\nu}\cdot\boldsymbol{u} = 0, \quad \operatorname{curl}\boldsymbol{u}\times\boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ \boldsymbol{u}(x,0) = \boldsymbol{b} & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

For the nonstationary Stokes equations, we already had the L^q-L^r estimates due to Iwashita, thus what we have to do here is to get the L^q-L^r estimates for the solutions of (1.1).

To state main results of this paper precisely, at this point we shall introduce notation used throughout this paper. We use the following symbols for denoting the special sets, $B_R = \{x \in \mathbb{R}^3 \mid |x| < R\}$, $S_R = \{x \in \mathbb{R}^3 \mid |x| = R\}$, $D_{L,R} = \{x \in \mathbb{R}^3 \mid L \leq |x| \leq R\}$, $\Omega_R = \Omega \cap B_R$, $\partial\Omega_R = \partial\Omega \cup S_R$.

Let D be any domain in $\mathbb{R}^3$. For $1 \leq q \leq \infty$, $L^q(D)$ denotes the usual Lebesgue space on D, $W^{m,q}(D)$ denotes the usual L^q-Sobolev space of order m, and $C_0^\infty(D)$ is the set of all infinitely differentiable functions in D with compact support in D. For function spaces of vector valued functions, we use the following symbols:

$$\boldsymbol{L}^q(D) = \{\boldsymbol{f} = (f_1, f_2, f_3) \mid f_j \in L^q(D), j = 1, 2, 3\},$$

likewise for $\boldsymbol{W}^{m,q}(D)$, $\boldsymbol{C}_0^\infty(D)$. Moreover we define a function space $\boldsymbol{L}^q_R(D)$ as follow:

$$\boldsymbol{L}^q_R(D) = \{\boldsymbol{f} \in \boldsymbol{L}^q(D) \mid \operatorname{supp}\boldsymbol{f} \subset B_R\}.$$

For the differentiation of three-vector of functions $\boldsymbol{f} = (f_1, f_2, f_3)$ and the scalar function p we use the following symbols: $\partial_j p = \partial p/\partial x_j$, $p_t = \partial_t p = \partial p/\partial t$, $\nabla p = (\partial_1 p, \partial_2 p, \partial_3 p)$,

$$\operatorname{div}\boldsymbol{f} = \sum_{j=1}^{3} \partial_j f_j, \quad \operatorname{curl}\boldsymbol{f} = (\partial_2 f_3 - \partial_3 f_2, \partial_3 f_1 - \partial_1 f_3, \partial_1 f_2 - \partial_2 f_1),$$
$$\nabla^m \boldsymbol{f} = (\partial_x^\alpha \boldsymbol{f} \mid |\alpha| = m).$$

To denote various constants, we use the same letters C and $C_{A,B,\dots}$ means that the constant depends on $A, B, \dots$. The constants C and $C_{A,B,\dots}$ may change from line to line.

In order to give an operator theoretic interpretation of (MHD), here we shall introduce the well known Helmholtz decomposition of $\boldsymbol{L}^q(\Omega)$. First, we shall introduce the following function space:

$$C_{0,\sigma}^\infty(\Omega) = \{\boldsymbol{f} \in \boldsymbol{C}_0^\infty(\Omega) \mid \operatorname{div}\boldsymbol{f} = 0 \text{ in } \Omega\}.$$

Let $1 < q < \infty$. As is well known that the Banach space $\boldsymbol{L}^q(\Omega)$ admits the Helmholtz decomposition (see Miyakawa [17], Galdi [6, Chapter III] and Simader and Sohr [21]):

$$\boldsymbol{L}^q(\Omega) = L^q_\sigma(\Omega) \oplus G^q(\Omega), \quad \oplus : \text{direct sum}.$$

Here

$$L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^q(\Omega)}},$$
$$G^q(\Omega) = \{\boldsymbol{f} \in \boldsymbol{L}^q(\Omega) \mid \boldsymbol{f} = \nabla p \text{ for some } p \in L^q_{\mathrm{loc}}(\overline{\Omega})\}.$$

Since $\partial\Omega$ is $C^{2,1}$-hypersurface, the solenoidal space $L^q_\sigma(\Omega)$ is characterized as (see e.g., Galdi [6])

$$L^q_\sigma(\Omega) = \{\boldsymbol{f} \in \boldsymbol{L}^q(\Omega) \mid \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega,\ \boldsymbol{\nu} \cdot \boldsymbol{f} = 0 \text{ on } \partial\Omega\}. \tag{1.2}$$

Let $P = P_{q,\Omega}$ be a continuous projection from $\boldsymbol{L}^q(\Omega)$ onto $L^q_\sigma(\Omega)$ and then

$$\|P\boldsymbol{f}\|_{L^q(\Omega)} \le C_q \|\boldsymbol{f}\|_{L^q(\Omega)} \tag{1.3}$$

for any $\boldsymbol{f} \in \boldsymbol{L}^q(\Omega)$. Let us define the linear operators $A = A_{q,\Omega}$ and $\mathcal{M} = \mathcal{M}_{q,\Omega}$ as follows:

$$\begin{aligned}
\mathcal{D}(A) &= L^q_\sigma(\Omega) \cap \boldsymbol{W}^{2,q}(\Omega) \cap \boldsymbol{W}^{1,q}_0(\Omega),\\
A\boldsymbol{v} &= -P\Delta\boldsymbol{v} \text{ for } \boldsymbol{v} \in \mathcal{D}(A),\\
\mathcal{D}(\mathcal{M}) &= L^q_\sigma(\Omega) \cap \{\boldsymbol{B} \in \boldsymbol{W}^{2,q}(\Omega) \mid \operatorname{curl} \boldsymbol{B} \times \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\},\\
\mathcal{M}\boldsymbol{B} &= \operatorname{curl}\operatorname{curl} \boldsymbol{B} \text{ for } \boldsymbol{B} \in \mathcal{D}(\mathcal{M}).
\end{aligned}$$

The operator A is usually called *the Stokes operator* with non slip boundary condition. We note that the operator $\mathcal{M}$ is mapping from $\mathcal{D}(\mathcal{M})$ to $L^q_\sigma(\Omega)$. By using A and $\mathcal{M}$, (MHD) is rewritten by the following Cauchy problem of abstract evolution equations in the Banach space $L^q_\sigma(\Omega) \times L^q_\sigma(\Omega)$:

$$\begin{cases}
\dfrac{d\boldsymbol{v}(t)}{dt} + A\boldsymbol{v}(t) + P[(\boldsymbol{v}(t)\cdot\nabla)\boldsymbol{v}(t) - (\boldsymbol{B}(t)\cdot\nabla)\boldsymbol{B}(t)] = 0, & t > 0,\\
\dfrac{d\boldsymbol{B}(t)}{dt} + \mathcal{M}\boldsymbol{B}(t) + (\boldsymbol{v}(t)\cdot\nabla)\boldsymbol{B}(t) - (\boldsymbol{B}(t)\cdot\nabla)\boldsymbol{v}(t) = 0, & t > 0,\\
\boldsymbol{v}(0) = \boldsymbol{a}, \quad \boldsymbol{B}(0) = \boldsymbol{b}.
\end{cases} \tag{ACP}$$

Here we have used the well known formula:

$$\boldsymbol{B} \times \operatorname{curl} \boldsymbol{B} = -(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \frac{\nabla|\boldsymbol{B}|^2}{2}.$$

The second term in the right hand side of the above relation is eliminated by the Helmholtz projection P. According to Miyakawa [17] and Borchers and Sohr [3], $-A$ generates a bounded analytic semigroup $(e^{-tA})_{t\ge0}$ on $L^q_\sigma(\Omega)$ and according to Miyakawa [16] and Shibata and Yamaguchi [20] the operator $-\mathcal{M}$ also generates a

bounded analytic semigroup $(e^{-t\mathcal{M}})_{t\geq 0}$ on $L^q_\sigma(\Omega)$. Therefore, by virtue of Duhamel's principle, (ACP) is converted into the following system of integral equations:

$$\begin{cases} \boldsymbol{v}(t) = e^{-tA}\boldsymbol{a} - \displaystyle\int_0^t e^{-(t-s)A}P[(\boldsymbol{v}(s)\cdot\nabla)\boldsymbol{v}(s) - (\boldsymbol{B}(s)\cdot\nabla)\boldsymbol{B}(s)]\,ds, \\ \boldsymbol{B}(t) = e^{-t\mathcal{M}}\boldsymbol{b} - \displaystyle\int_0^t e^{-(t-s)\mathcal{M}}[(\boldsymbol{v}(s)\cdot\nabla)\boldsymbol{B}(s) - (\boldsymbol{B}(s)\cdot\nabla)\boldsymbol{v}(s)]\,ds. \end{cases} \tag{INT}$$

For notational simplicity, we set $\boldsymbol{v}_0(t) = e^{-tA}\boldsymbol{a}$, $\boldsymbol{B}_0(t) = e^{-t\mathcal{M}}\boldsymbol{b}$,

$$F[\boldsymbol{v},\boldsymbol{B}](t) = -\int_0^t e^{-(t-s)A}P[(\boldsymbol{v}(s)\cdot\nabla)\boldsymbol{v}(s) - (\boldsymbol{B}(s)\cdot\nabla)\boldsymbol{B}(s)]\,ds,$$
$$G[\boldsymbol{v},\boldsymbol{B}](t) = -\int_0^t e^{-(t-s)\mathcal{M}}[(\boldsymbol{v}(s)\cdot\nabla)\boldsymbol{B}(s) - (\boldsymbol{B}(s)\cdot\nabla)\boldsymbol{v}(s)]\,ds.$$

Our aim of this paper is deduced to solve (INT) by contraction mapping principle (or Kato's iteration scheme). In order to do this, we need L^q-L^r estimates for the semigroups e^{-tA} and $e^{-t\mathcal{M}}$.

We are now in a position to state our main results. The first result is concerning L^q-L^r estimates for the semigroup $e^{-t\mathcal{M}}$.

Theorem 1.1 (L^q-L^r estimates).
(i) *Let $1 \leq q \leq r \leq \infty$ and $q \neq \infty, r \neq 1$. Then there exists a constant $C = C_{q,r} > 0$ such that*

$$\|e^{-t\mathcal{M}}\boldsymbol{f}\|_{L^r(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\boldsymbol{f}\|_{L^q(\Omega)}, \quad t > 0$$

for any $\boldsymbol{f} \in L^q_\sigma(\Omega)$.
(ii) *Let $1 \leq q \leq r \leq 3$, $r \neq 1$. Then there exists a constant $C = C_{q,r} > 0$ such that*

$$\|\nabla e^{-t\mathcal{M}}\boldsymbol{f}\|_{L^r(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)}, \quad t > 0$$

for any $\boldsymbol{f} \in L^q_\sigma(\Omega)$.

The basic idea to prove Theorem 1.1 is similar to that of Iwashita [10] for the Stokes semigroup. Iwashita's idea is based on the local energy decay property of the semigroup near the obstacle $\mathcal{O}$. Such local energy decay estimate for $e^{-t\mathcal{M}}$ is obtained by Shibata and Yamaguchi [20] (see also [23]).

Theorem 1.2 (local energy decay [20]). *Let $1 < q < \infty$. For any $R > R_0$, there exists a constant $C = C_{q,R} > 0$ such that*

$$\|e^{-t\mathcal{M}}\boldsymbol{f}\|_{W^{2,q}(\Omega_R)} \leq Ct^{-\frac{3}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)}, \quad t \geq 1,$$

for any $\boldsymbol{f} \in L^q_\sigma(\Omega) \cap \boldsymbol{L}^q_R(\Omega)$.

The following theorem by Iwashita [10] is concerning the L^q-L^r estimates for the Stokes semigroup, which is refined by Maremonti and Solonnikov [15] and Enomoto and Shibata [5] (see also Giga and Sohr [8]).

Theorem 1.3 (L^q-L^r estimates for the Stoke semigroup [5, 8, 10, 15]).
(i) *Let $1 \le q \le r \le \infty$ and $q \neq \infty, r \neq 1$. Then there exists a constant $C = C_{q,r} > 0$ such that*

$$\|e^{-tA}\boldsymbol{f}\|_{L^r(\Omega)} \le Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\boldsymbol{f}\|_{L^q(\Omega)}, \quad t > 0$$

for any $\boldsymbol{f} \in L^q_\sigma(\Omega)$.
(ii) *Let $1 < q \le r \le 3$. Then there exists a constant $C = C(q, r) > 0$ such that*

$$\|\nabla e^{-tA}\boldsymbol{f}\|_{L^r(\Omega)} \le Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)}, \quad t > 0$$

for any $\boldsymbol{f} \in L^q_\sigma(\Omega)$.

Finally, applying Theorem 1.1 and Theorem 1.3 we obtain an existence theorem of global in time strong solution for (MHD) with small initial data.

Theorem 1.4 (Global existence). *There exists an $\eta = \eta(\Omega) > 0$ such that if $(\boldsymbol{a}, \boldsymbol{b}) \in L^3_\sigma(\Omega) \times L^3_\sigma(\Omega)$ satisfies $\|(\boldsymbol{a}, \boldsymbol{b})\|_3 \le \eta$ then* (MHD) *has a unique global strong solution $(\boldsymbol{v}(t), \boldsymbol{B}(t)) \in BC([0, \infty); L^3_\sigma(\Omega) \times L^3_\sigma(\Omega))$ which possesses the followings*:

$$\begin{aligned}
&\lim_{t\to 0}\|(\boldsymbol{v}(t), \boldsymbol{B}(t) - (\boldsymbol{a}, \boldsymbol{b}))\|_{L^3(\Omega)} = 0,\\
&\lim_{t\to +0} t^{\frac{1}{2}-\frac{3}{2q}}\|(\boldsymbol{v}(t), \boldsymbol{B}(t))\|_{L^q(\Omega)}\\
&\quad + \lim_{t\to +0} t^{\frac{1}{2}}\|\nabla(\boldsymbol{v}(t), \boldsymbol{B}(t))\|_{L^3(\Omega)} = 0 \quad \textit{for } 3 < q < \infty;\\
&\|(\boldsymbol{v}(t), \boldsymbol{B}(t))\|_{L^q(\Omega)} = o\left(t^{-\frac{1}{2}+\frac{3}{2q}}\right) \quad \textit{for } 3 \le q \le \infty, && (1.4)\\
&\|\nabla(\boldsymbol{v}(t), \boldsymbol{B}(t))\|_{L^3(\Omega)} = o\left(t^{-\frac{1}{2}}\right). && (1.5)
\end{aligned}$$

as $t \to \infty$. Here $BC(I; X)$ denotes the class of X-valued bounded and continuous function on interval I.

Remark 1.5. We do not require any smallness assumption on the initial data for proving the local in time existence of solution to (MHD).

Below, in section 2 we prepare the well known Bogovskiĭ's lemma and some lemmas which will be used in the latter sections. In section 3 we shall prove Theorem 1.1 with aid of L^q-L^r estimates for the heat kernel, Theorem 1.2 and cut-off technique. By using Theorems 1.1 and 1.3, we prove Theorem 1.4 in section 4.

2 Preliminaries

In this section, we prepare some useful lemmas which will be used in the latter sections. In Section 3 we will prove Theorem 1.1 by cut-off technique. In order to keep the divergence free condition in cut-off procedure, we are due to the well known lemma by Bogovskiĭ [2] (see also Galdi [6, Chapter III]). In order to state Bogovskiĭ's lemma, we shall introduce the function spaces $\dot{W}^{m,q}(D)$ and $\dot{W}^{m,q}_a(D)$ as follows:

$$\dot{W}^{m,q}(D) = \overline{C_0^\infty(D)}^{\|\cdot\|_{W^{m,q}}},$$
$$\dot{W}^{m,q}_a(D) = \left\{ f \in \dot{W}^{m,q}(D) \,\middle|\, \int_D f(x)\,dx = 0 \right\}.$$

Here D stands for a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂D. We note that $\dot{W}^{0,q}(D) = L^q(D)$.

Lemma 2.1. *Let $1 < q < \infty$ and let m be a non-negative integer. Then there exists a bounded linear operator $\mathbb{B} \equiv \mathbb{B}_D : \dot{W}^{m,q}_a(D) \to \dot{\boldsymbol{W}}^{m+1,q}(\mathbb{R}^3)$ such that*

$$\operatorname{supp} \mathbb{B}[f] \subset D,$$
$$\operatorname{div} \mathbb{B}[f] = f \ \text{ in } \mathbb{R}^3.$$

To use Lemma 2.1, we shall rely on the following lemma.

Lemma 2.2. *Let $1 < q < \infty$, $R > L > R_0$ and let $\varphi(x) \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \le L$ and $\varphi(x) = 0$ for $|x| \ge R$.*

(i) *If $\boldsymbol{u} \in \boldsymbol{W}^{2,q}(\mathbb{R}^3)$ and $\boldsymbol{u}$ satisfies the condition*: $\operatorname{div} \boldsymbol{u} = 0$ *in $\mathbb{R}^3$, then $(\nabla\varphi) \cdot \boldsymbol{u} \in \dot{W}^{2,q}_a(D_{L,R})$.*

(ii) *If $\boldsymbol{u} \in \boldsymbol{W}^{2,q}(\Omega)$ and $\boldsymbol{u}$ satisfies the conditions*: $\operatorname{div} \boldsymbol{u} = 0$ *in Ω and $\boldsymbol{\nu} \cdot \boldsymbol{u} = 0$ on $\partial\Omega$, then $(\nabla\varphi) \cdot \boldsymbol{u} \in \dot{W}^{2,q}_a(D_{L,R})$.*

Next, we shall introduce the results in the case of bounded domain D. From Akiyama, Kasai, Shibata and Tsutsumi [1], it follows the following proposition.

Proposition 2.3. *Let $1 < q < \infty$. Assume that $\partial D \in C^{2,1}$. Then for any $\boldsymbol{f} \in \boldsymbol{L}^q(D)$ there exists a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{2,q}(D)$ of the following system*:

$$\begin{cases} \boldsymbol{u} - \Delta\boldsymbol{u} = \boldsymbol{f} & \text{in} \quad D, \\ \operatorname{curl} \boldsymbol{u} \times \boldsymbol{\nu} = 0 & \text{on} \quad \partial D, \\ \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 & \text{on} \quad \partial D, \end{cases}$$

which satisfies the estimate:

$$\|\boldsymbol{u}\|_{W^{2,q}(D)} \le C\|\boldsymbol{f}\|_{L^q(D)}.$$

Next we shall introduce the resolvent estimate. The resolvent problem corresponding to (1.1) is given by the following Laplace system:

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f} & \text{in} \quad \Omega, \\ \operatorname{curl} \boldsymbol{u} \times \boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega, \\ \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

The following theorem obtained by Akiyama, Kasai, Shibata and Tsutsumi [1] is concerned with the resolvent estimate for (2.1).

Theorem 2.4. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $\delta > 0$. Set*

$$\Sigma_{\epsilon,\delta} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \le \pi - \epsilon, |\lambda| \ge \delta\}.$$

Then, for any $\boldsymbol{f} \in L^q_\sigma(\Omega)$ and $\lambda \in \Sigma_{\epsilon,\delta}$, (2.1) admits a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{2,q}(\Omega)$ possessing the estimate:

$$|\lambda| \|\boldsymbol{u}\|_{L^q(\Omega)} + \|\boldsymbol{u}\|_{W^{2,q}(\Omega)} \le C_{\epsilon,\delta} \|\boldsymbol{f}\|_{L^q(\Omega)}. \tag{2.2}$$

On the linear operator $\mathcal{M}_q$ defined in Section 1, we quote the following theorem due to Shibata and Yamaguchi [20].

Theorem 2.5. *Let $1 < q < \infty$, $q' = q/(q-1)$ and $\mathcal{M}_q^*$ be an adjoint operator of $\mathcal{M}_q$. Then we have $\mathcal{M}_q^* = \mathcal{M}_{q'}$.*

3 Proof of Theorem 1.1

In this section we shall prove Theorem 1.1. Our proof is based on the ideas due to Iwashita [10] and Hishida [9]. Here and hereafter $T(t)$ denotes the analytic semigroup generated by $-\mathcal{M}_q$, i.e., $T(t) \equiv e^{-t\mathcal{M}}$. Given $\boldsymbol{f} \in L^q_\sigma(\Omega)$, we set $\boldsymbol{u}(t) = T(t)\boldsymbol{f}$. Then $\boldsymbol{u}(t)$ solves the following initial-boundary value problem:

$$\begin{cases} \boldsymbol{u}_t - \Delta \boldsymbol{u} = 0, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \boldsymbol{\nu} \cdot \boldsymbol{u} = 0, \quad \operatorname{curl} \boldsymbol{u} \times \boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ \boldsymbol{u}(x,0) = \boldsymbol{f} & \text{in} \quad \Omega. \end{cases} \tag{3.1}$$

Here we have used the well known formula:

$$\Delta \boldsymbol{u} = \nabla \operatorname{div} \boldsymbol{u} - \operatorname{curl} \operatorname{curl} \boldsymbol{u}. \tag{3.2}$$

1st step

As a first step, we shall show the following lemma.

Lemma 3.1. *Let $1 < q < \infty$ and $R > R_0 + 3$. Then there exists a $C = C_{q,\Omega,R} > 0$ such that*

$$\|\partial_t T(t)\boldsymbol{f}\|_{W^{1,q}(\Omega_R)} + \|T(t)\boldsymbol{f}\|_{W^{2,q}(\Omega_R)} \leq Ct^{-\frac{3}{2q}}\|\boldsymbol{f}\|_{L^q(\Omega)}$$

for any $t \geq 2$ and $\boldsymbol{f} \in L^q_\sigma(\Omega)$.

Proof. Since we consider the case when $t \geq 2$, we set

$$\boldsymbol{g} = T(1)\boldsymbol{f}, \quad \boldsymbol{v}(t) = T(t)\boldsymbol{g} = T(t+1)\boldsymbol{f}. \tag{3.3}$$

By (2.2), the analytic semigroup theory (see e.g., Pazy [18]) and (3.1), we have

$$\|\boldsymbol{g}\|_{W^{2,q}(\Omega)} \leq C\|\boldsymbol{f}\|_{L^q(\Omega)}, \quad \boldsymbol{g} \in \mathcal{D}(\mathcal{M}), \tag{3.4}$$

$$\boldsymbol{v}(t) \in C([0,\infty); \boldsymbol{W}^{2,q}(\Omega)) \cap C^1((0,\infty); \boldsymbol{L}^q(\Omega)), \tag{3.5}$$

$$\begin{cases} \boldsymbol{v}_t - \Delta\boldsymbol{v} = 0, \quad \operatorname{div}\boldsymbol{v} = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \boldsymbol{\nu}\cdot\boldsymbol{v} = 0, \quad \operatorname{curl}\boldsymbol{v}\times\boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ \boldsymbol{v}(x,0) = \boldsymbol{g} & \text{in} \quad \Omega. \end{cases} \tag{3.6}$$

Let $\psi \in C^\infty(\mathbb{R}^3)$ such that $\psi(x) = 1$ for $|x| \geq R+1$ and $\psi(x) = 0$ for $|x| \leq R$. By (3.4) and Lemma 2.2, we have $(\nabla\psi)\cdot\boldsymbol{g} \in W^{2,q}_a(D_{R,R+1})$ and therefore by Lemma 2.1 we have

$$\begin{gathered} \mathbb{B}_{D_{R,R+1}}[(\nabla\psi)\cdot\boldsymbol{g}] \in W^{3,q}(\mathbb{R}^3), \quad \operatorname{supp}\mathbb{B}_{D_{R,R+1}}[(\nabla\psi)\cdot\boldsymbol{g}] \subset D_{R,R+1}, \\ \operatorname{div}\mathbb{B}_{D_{R,R+1}}[(\nabla\psi)\cdot\boldsymbol{g}] = (\nabla\psi)\cdot\boldsymbol{g}, \\ \|\mathbb{B}_{D_{R,R+1}}[(\nabla\psi)\cdot\boldsymbol{g}]\|_{W^{3,q}(\mathbb{R}^3)} \leq C\|\boldsymbol{f}\|_{L^q(\Omega)}. \end{gathered} \tag{3.7}$$

In what follows, for notational simplicity, we use the abbreviation $\mathbb{B} = \mathbb{B}_{D_{R,R+1}}$.

Let $E(t)$ be the Gaussian kernel, namely,

$$E(t) = E(x,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}}\exp\left(-\frac{|x|^2}{4t}\right) \tag{3.8}$$

and set

$$\boldsymbol{h} = \psi\boldsymbol{g} - \mathbb{B}[(\nabla\psi)\cdot\boldsymbol{g}], \quad \boldsymbol{w} = E(t)*\boldsymbol{h} = \frac{1}{(4\pi t)^{3/2}}\int_{\mathbb{R}^3}\exp\left(-\frac{|x-y|^2}{4t}\right)\boldsymbol{h}(y)\,dy.$$

By (3.4) and (3.7), we see that

$$\begin{gathered} \boldsymbol{h} \in \boldsymbol{W}^{2,q}(\mathbb{R}^3), \\ \operatorname{div}\boldsymbol{h} = 0 \text{ in } \mathbb{R}^3, \\ \boldsymbol{h} = \boldsymbol{g}, \quad |x| \geq R+1, \\ \|\boldsymbol{h}\|_{W^{2,q}(\mathbb{R}^3)} \leq C_q\|\boldsymbol{f}\|_{L^q(\Omega)}. \end{gathered} \tag{3.9}$$

Applying Young's inequality to $\boldsymbol{w}(t)$ and using (3.9), we obtain

$$\boldsymbol{w}(t) \in C([0,\infty); \boldsymbol{W}^{2,q}(\mathbb{R}^3)) \cap C^1([0,\infty); \boldsymbol{L}^q(\mathbb{R}^3)), \tag{3.10}$$

$$\boldsymbol{w}_t - \Delta \boldsymbol{w} = 0, \quad \operatorname{div} \boldsymbol{w} = 0 \quad \text{in } \mathbb{R}^3 \times (0,\infty), \quad \boldsymbol{w}(0) = \boldsymbol{h}, \tag{3.11}$$

$$\begin{aligned} &\|\nabla^j \boldsymbol{w}(t)\|_{L^r(\mathbb{R}^3)} \le C_{q,r}(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{j}{2}} \|\boldsymbol{f}\|_{L^q(\Omega)}, \quad j=1,2,\ t \ge 1, \\ &\|\boldsymbol{w}_t\|_{L^r(\mathbb{R}^3)} + \|\nabla^2 \boldsymbol{w}(t)\|_{L^r(\mathbb{R}^3)} \le C_{q,r}(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-1} \|\boldsymbol{f}\|_{L^q(\Omega)} \end{aligned} \tag{3.12}$$

provided that $1 < q \le r \le \infty$. Since $\operatorname{div} \boldsymbol{w} = 0$, by Lemma 2.1 we have $(\nabla\psi)\cdot\boldsymbol{w}(t) \in C([0,\infty); \dot{W}^{2,q}_a(D_{R,R+1}))$, and therefore we set

$$\boldsymbol{z}(t) = \boldsymbol{v}(t) - \psi \boldsymbol{w}(t) + \mathbb{B}[(\nabla\psi)\cdot\boldsymbol{w}(t)]. \tag{3.13}$$

Then, from (3.5) and (3.10) and Lemma 2.1 we obtain

$$\boldsymbol{z}(t) \in C([0,\infty); \boldsymbol{W}^{2,q}(\Omega)) \cap C^1([0,\infty); \boldsymbol{L}^q(\Omega)), \tag{3.14}$$

$$\begin{cases} \boldsymbol{z}_t - \Delta \boldsymbol{z} = \boldsymbol{F}(t), \quad \operatorname{div} \boldsymbol{z} = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \boldsymbol{\nu}\cdot\boldsymbol{z} = 0, \quad \operatorname{curl} \boldsymbol{z} \times \boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ \boldsymbol{z}(0) = \boldsymbol{z}_0 & \text{in} \quad \Omega, \end{cases} \tag{3.15}$$

where we have set

$$\begin{gathered} \boldsymbol{F}(t) = 2\nabla \boldsymbol{w}(t)\cdot\nabla\psi + (\Delta\psi)\boldsymbol{w}(t) + (\partial_t - \Delta)\mathbb{B}[(\nabla\psi)\cdot\boldsymbol{w}(t)], \\ \boldsymbol{z}_0 = \boldsymbol{g} - \psi\boldsymbol{h} + \mathbb{B}[(\nabla\psi)\cdot\boldsymbol{h}]. \end{gathered} \tag{3.16}$$

We shall show that

$$\boldsymbol{F}(t) \in C([0,\infty); L^q_\sigma(\Omega)), \quad \operatorname{supp} \boldsymbol{F}(t) \subset D_{R,R+1} \quad \text{for any } t > 0, \tag{3.17}$$

$$\|\boldsymbol{F}(t)\|_{L^q(\Omega)} \le C(1+t)^{-\frac{3}{2q}} \|\boldsymbol{f}\|_{L^q(\Omega)}, \tag{3.18}$$

$$\boldsymbol{z}_0 \in \mathcal{D}(\mathcal{M}_q), \quad \boldsymbol{z}_0 = 0 \quad \text{for } x \notin B_{R+1}, \tag{3.19}$$

$$\|\boldsymbol{z}_0\|_{W^{2,q}(\Omega)} \le C\|\boldsymbol{f}\|_{L^q(\Omega)}. \tag{3.20}$$

In fact, since

$$(\partial_t - \Delta)(\psi\boldsymbol{w}(t)) = -2\nabla\boldsymbol{w}(t)\cdot\nabla\psi - (\Delta\psi)\boldsymbol{w}(t),$$

by Lemma 2.2 we have

$$\begin{aligned} \operatorname{div} \boldsymbol{F}(t) &= -\operatorname{div}\{(\partial_t - \Delta)(\psi\boldsymbol{w}(t))\} + (\partial_t - \Delta)\operatorname{div}\mathbb{B}[(\nabla\psi)\cdot\boldsymbol{w}(t)] \\ &= -(\partial_t - \Delta)[\operatorname{div}(\psi\boldsymbol{w}(t)) - (\nabla\psi)\cdot\boldsymbol{w}(t)] = 0, \end{aligned} \tag{3.21}$$

because $\operatorname{div} \boldsymbol{w}(t) = 0$. Obviously, $\operatorname{supp} \boldsymbol{F}(t) \subset D_{R,R+1}$. In particular, we have $\boldsymbol{\nu}\cdot\boldsymbol{F}(t) = 0$ on $\partial\Omega$ for any $t \ge 0$, which combined with (1.2) and (3.21) implies that

$\boldsymbol{F}(t) \in L^q_\sigma(\Omega)$ for any $t \geq 0$. Clearly, by (3.5) and (3.14), $\boldsymbol{F}(t) \in C([0,\infty); \boldsymbol{L}^q(\Omega))$, which completes the proof of (3.17). By Lemma 2.1 and (3.12) with $r = \infty$, we have

$$\begin{aligned}
\|\boldsymbol{F}(t)\|_{L^q(\Omega)} &\leq C_q\{\||\nabla\psi|\nabla \boldsymbol{w}(t)\|_{L^q(\Omega)} + \||\Delta\psi|\boldsymbol{w}(t)\|_{L^q(\Omega)} \\
&\qquad + \|\nabla\psi\cdot\boldsymbol{w}(t)\|_{W^{1,q}(\Omega)} + \|\nabla\psi\cdot\boldsymbol{w}_t(t)\|_{L^q(\Omega)}\} \\
&\leq C_{q,R}\{\|\boldsymbol{w}(t)\|_{W^{1,\infty}(\mathbb{R}^3)} + \|\boldsymbol{w}_t(t)\|_{L^\infty(\mathbb{R}^3)}\} \\
&\leq C_{q,R}(1+t)^{-\frac{3}{2q}}\|\boldsymbol{f}\|_{L^q(\Omega)}.
\end{aligned}$$

By (3.9) we see that $\boldsymbol{g} = \psi\boldsymbol{h}$ for $x \notin B_{R+1}$. Furthermore, $\operatorname{supp}\mathbb{B}[(\nabla\psi)\cdot\boldsymbol{h}] \subset D_{R,R+1}$. Therefore, by (3.9) we have $\operatorname{div}\boldsymbol{z}_0 = 0$ in Ω, $\|\boldsymbol{z}_0\|_{W^{2,q}(\Omega)} \leq C_q\|\boldsymbol{f}\|_{L^q(\Omega)}$ and $\boldsymbol{z}_0 = 0$ for $x \notin B_{R+1}$. Since $\boldsymbol{z}_0 = \boldsymbol{g}$ for $|x| \leq R$, $\boldsymbol{g} = T(1)\boldsymbol{f}$ implies that $\boldsymbol{\nu}\cdot\boldsymbol{z}_0 = 0$ and $\operatorname{curl}\boldsymbol{z}_0 \times \boldsymbol{\nu} = 0$ on $\partial\Omega$. These facts imply that $\boldsymbol{z}_0 \in \mathcal{D}(\mathcal{M}_q)$. Therefore we get (3.17), (3.18) and (3.19).

By (3.14), (3.15) and (3.17) and Duhamel's principle, we have

$$\boldsymbol{z}(t) = T(t)\boldsymbol{z}_0 + \int_0^t T(t-s)\boldsymbol{F}(s)\,ds. \tag{3.22}$$

Let $t \geq 1$. In view of (3.17) and (3.19), we can apply Theorem 1.2 (local energy decay) to estimate $\boldsymbol{z}(t)$, and then we have

$$\begin{aligned}
\|\boldsymbol{z}\|_{W^{1,q}(\Omega_R)} \leq C_R t^{-\frac{3}{2}}\|\boldsymbol{z}_0\|_{L^q(\Omega)} &+ \int_{t-1}^t (t-s)^{-\frac{1}{2}}\|\boldsymbol{F}(s)\|_{L^q(\Omega)}\,ds \\
&+ \int_0^{t-1}(t-s)^{-\frac{3}{2}}\|\boldsymbol{F}(s)\|_{L^q(\Omega)}\,ds.
\end{aligned} \tag{3.23}$$

Here we have used the standard estimate of analytic semigroup:

$$\|T(t)\boldsymbol{f}\|_{W^{1,q}(\Omega)} \leq Ct^{-\frac{1}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)}$$

for any $0 < t \leq 1$ and $\boldsymbol{f} \in L^q_\sigma(\Omega)$, which follows from (2.2). By using (3.18), (3.20) and (3.23) we obtain

$$\|\boldsymbol{z}(t)\|_{W^{1,q}(\Omega_{R+1})} \leq Ct^{-\frac{3}{2q}}\|\boldsymbol{f}\|_{L^q(\Omega)} \quad t \geq 1. \tag{3.24}$$

Applying (3.12) with $r = \infty$ and Lemma 2.1 we have

$$\begin{aligned}
&\|\psi\boldsymbol{w}(t)\|_{W^{1,q}(\Omega_{R+1})} \leq C_q\|\boldsymbol{w}(t)\|_{W^{1,\infty}(\mathbb{R}^3)} \leq C_q t^{-\frac{3}{2q}}\|\boldsymbol{f}\|_{L^q(\Omega)}, \\
&\|\mathbb{B}[(\nabla\psi)\cdot\boldsymbol{w}(t)]\|_{W^{1,q}(\Omega_{R+1})} \leq C_q\|(\nabla\psi)\cdot\boldsymbol{w}(t)\|_{L^q(\Omega_{R+1})} \leq C_q\|\boldsymbol{w}(t)\|_{L^\infty(\mathbb{R}^3)} \\
&\qquad\qquad \leq C_q t^{-\frac{3}{2q}}\|\boldsymbol{f}\|_{L^q(\Omega)},
\end{aligned}$$

which combined with (3.13) and (3.24) implies that

$$\|\boldsymbol{v}(t)\|_{W^{1,q}(\Omega_{R+1})} \le C_q t^{-\frac{3}{2q}} \|\boldsymbol{f}\|_{L^q(\Omega)} \quad \text{for any } t \ge 1. \tag{3.25}$$

Now, we shall estimate $\partial_t \boldsymbol{v}(t)$. Recalling that $\boldsymbol{v}(t) = T(t+1)\boldsymbol{f} \in C^2([0,\infty); \mathcal{D}(\mathcal{M}))$, differentiating (3.6) with respect to t variable, we have

$$\begin{cases} \partial_t \boldsymbol{v}_t - \Delta \boldsymbol{v}_t = 0, \quad \operatorname{div} \boldsymbol{v}_t = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \boldsymbol{\nu} \cdot \boldsymbol{v}_t = 0, \quad \operatorname{curl} \boldsymbol{v}_t \times \boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ \boldsymbol{v}_t|_{t=0} = \boldsymbol{g}' & \text{in} \quad \Omega, \end{cases} \tag{3.26}$$

where $\boldsymbol{g}' = \partial_t T(t+1)\boldsymbol{f}|_{t=0}$. Since $\boldsymbol{g}' \in \mathcal{D}(\mathcal{M})$ and $\|\boldsymbol{g}'\|_{W^{2,q}(\Omega)} \le C_q \|\boldsymbol{f}\|_{L^q(\Omega)}$, applying the same argument as above to (3.26), we get

$$\|\partial_t \boldsymbol{v}(t)\|_{W^{1,q}(\Omega_{R+1})} \le C_q t^{-\frac{3}{2q}} \|\boldsymbol{f}\|_{L^q(\Omega)}. \tag{3.27}$$

Finally we shall estimate the second derivative of $\boldsymbol{v}(t)$. In order to do this, we shall use Theorem 2.4 with $\lambda = 1$. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \le R$ and $\varphi(x) = 0$ for $|x| \ge R + 1/2$. Put

$$\boldsymbol{v}_1(t) = \varphi \boldsymbol{v}(t) - \mathbb{B}[(\nabla\varphi) \cdot \boldsymbol{v}(t)].$$

Here and in the followings, we use the abbreviations $\mathbb{B} \equiv \mathbb{B}_{D_{R+1/2,R}}$. By Lemma 2.1 and Lemma 2.2 we have

$$\boldsymbol{v}_1(t) = \boldsymbol{v}(t) \quad \text{in } \Omega_R, \quad \operatorname{div} \boldsymbol{v}_1(t) = 0 \quad \text{in } \Omega. \tag{3.28}$$

According to (3.6), (3.28) and the fact that $\boldsymbol{v}_1(t) = 0$ for $x \notin B_{R+1/2}$, we have

$$\begin{cases} \boldsymbol{v}_1(t) - \Delta \boldsymbol{v}_1(t) = \boldsymbol{G}(t), \quad \operatorname{div} \boldsymbol{v}_1 = 0 & \text{in} \quad \Omega_{R+1} \times (0,\infty), \\ \boldsymbol{\nu} \cdot \boldsymbol{v}_1(t) = 0, \quad \operatorname{curl} \boldsymbol{v}_1(t) \times \boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega_{R+1} \times (0,\infty), \end{cases}$$

where

$$\boldsymbol{G}(t) = \varphi \boldsymbol{v}(t) - \mathbb{B}[(\nabla\varphi) \cdot \boldsymbol{v}(t)] - 2\nabla \boldsymbol{v}(t) \cdot \nabla\varphi - (\Delta\varphi)\boldsymbol{v}(t) + \Delta\mathbb{B}[(\nabla\varphi) \cdot \boldsymbol{v}(t)] + \varphi \partial_t \boldsymbol{v}(t).$$

By Proposition 2.3 we have

$$\|\boldsymbol{v}_1(t)\|_{W^{2,q}(\Omega_{R+1})} \le \|\boldsymbol{G}(t)\|_{L^q(\Omega_{R+1})}. \tag{3.29}$$

Applying (3.25) and (3.27), we have

$$\|\boldsymbol{G}(t)\|_{L^q(\Omega_{R+1})} \le C_q t^{-\frac{3}{2q}} \|\boldsymbol{f}\|_{L^q(\Omega)},$$

which combined with (3.28) and (3.29) implies that

$$\|\boldsymbol{v}(t)\|_{W^{2,q}(\Omega_R)} \le C_{q,R} t^{-\frac{3}{2q}} \|\boldsymbol{f}\|_{L^q(\Omega)}. \tag{3.30}$$

Combining (3.3), (3.27) and (3.30), we complete the proof of the lemma. $\square$

2nd step

At this step, we shall show the following lemma.

Lemma 3.2. *Let* $1 < q < \infty$ *and* $\boldsymbol{f} \in L^q_\sigma(\Omega)$. *Then we have the following two estimates*:

$$\|T(t)\boldsymbol{f}\|_{L^r(\Omega)} \le C_{q,r} t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\boldsymbol{f}\|_{L^q(\Omega)} \quad \textit{for any } t \ge 2 \tag{3.31}$$

provided that $q \le r \le \infty$ *and* $3(1/q - 1/r) < 2$ *and*

$$\|\nabla T(t)\boldsymbol{f}\|_{L^q(\Omega)} \le C_q t^{-\frac{1}{2}} \|\boldsymbol{f}\|_{L^q(\Omega)} \quad \textit{for any } t \ge 2 \tag{3.32}$$

provided that $1 < q \le 3$.

Proof. In view of Lemma 3.1, it suffices to estimate $T(t)\boldsymbol{f}$ in $\Omega \setminus B_R$ for $t \ge 2$. Set $\boldsymbol{v}(t) = T(t+1)\boldsymbol{f} = T(t)\boldsymbol{g}$ with $\boldsymbol{g} = T(1)\boldsymbol{f}$. Let $\varphi(x) \in C^\infty(\mathbb{R}^3)$ so that $\varphi(x) = 1$ for $|x| \ge R-1$ and $\varphi(x) = 0$ for $|x| \le R-2$. In view of Lemma 2.2, we set $\boldsymbol{w}(t) = \varphi\boldsymbol{v}(t) - \mathbb{B}[(\nabla\varphi)\cdot\boldsymbol{v}(t)]$ and then by (3.6) and Lemma 2.1 we have

$$\begin{cases} \boldsymbol{w}_t - \Delta\boldsymbol{w} = \boldsymbol{K}(t), \quad \operatorname{div}\boldsymbol{w} = 0 \qquad \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ \boldsymbol{w}(0) = \boldsymbol{w}_0 \end{cases} \tag{3.33}$$

where

$$\begin{gathered} \boldsymbol{K}(t) = -2\nabla\boldsymbol{v}(t)\cdot\nabla\varphi(t) - (\Delta\varphi)\boldsymbol{v}(t) - (\partial_t - \Delta)\mathbb{B}[(\nabla\varphi)\cdot\boldsymbol{v}(t)], \\ \boldsymbol{w}_0 = \varphi\boldsymbol{g} - \mathbb{B}[(\nabla\varphi)\cdot\boldsymbol{g}]. \end{gathered} \tag{3.34}$$

Here and hereafter $\mathbb{B} \equiv \mathbb{B}_{R-2,R-1}$. Since $\boldsymbol{w}(t) = \boldsymbol{v}(t)$ for $|x| \ge R$, it suffices to estimate (3.34). Employing the same arguments as in the proof of (3.17) and (3.19), we get

$$\operatorname{div}\boldsymbol{K}(t) = 0, \operatorname{div}\boldsymbol{w}_0 = 0 \quad \text{in } \mathbb{R}^3, \tag{3.35}$$

$$\operatorname{supp}\boldsymbol{K}(t) \subset D_{R-2,R-1}. \tag{3.36}$$

Let $E(t)$ be the Gaussian kernel: (3.8). In view of (3.35), employing the same argument as is the proof of (3.22), we have

$$\boldsymbol{w}(t) = E(t) * \boldsymbol{w}_0 + \int_0^t E(t-s) * \boldsymbol{K}(s)\,ds. \tag{3.37}$$

Applying Young's inequality, we have

$$\|\nabla^j E(t) * \varphi\|_{L^r(\mathbb{R}^3)} \le C_{q,r} t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{j}{2}} \|\varphi\|_{L^q(\mathbb{R}^3)} \tag{3.38}$$

for any $t > 0$, $j \geq 0$ and $1 \leq q \leq r \leq \infty$, and

$$\|E(t) * \varphi\|_{W^{2,q}(\mathbb{R}^3)} \leq Ct^{-\frac{1}{2}} \|\varphi\|_{W^{1,q}(\mathbb{R}^3)} \tag{3.39}$$

for $0 < t \leq 2$. Recalling that $\boldsymbol{v}(t) = T(t+1)\boldsymbol{f}$, by (2.2) we have

$$\|\partial_t \boldsymbol{v}(t)\|_{L^q(\Omega)} + \|\boldsymbol{v}(t)\|_{W^{2,q}(\Omega)} \leq C_q \|\boldsymbol{f}\|_{L^q(\Omega)} \tag{3.40}$$

for $0 < t \leq 2$. From (3.4), (3.36), (3.40), Lemma 2.1 and Lemma 3.1, we have

$$\|\boldsymbol{K}(t)\|_{W^{1,q}(\mathbb{R}^3)} + \|\boldsymbol{K}(t)\|_{L^\gamma(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2q}} \|\boldsymbol{f}\|_{L^q(\Omega)}, \quad 1 \leq \gamma \leq q, \tag{3.41}$$

$$\|\boldsymbol{w}_0\|_{L^q(\mathbb{R}^3)} \leq C \|\boldsymbol{f}\|_{L^q(\Omega)}. \tag{3.42}$$

Set

$$I_1(t) = E(t) * \boldsymbol{w}_0, \quad I_2(t) = \int_0^t E(t-s) * \boldsymbol{K}(s)\, ds.$$

By (3.38) and (3.42) we have

$$\begin{aligned} \|I_1(t)\|_{L^r(\mathbb{R}^3)} &\leq C_{q,r} t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\boldsymbol{f}\|_{L^q(\Omega)}, \\ \|\nabla I_1(t)\|_{L^q(\mathbb{R}^3)} &\leq C_q t^{-\frac{1}{2}} \|\boldsymbol{f}\|_{L^q(\Omega)}. \end{aligned} \tag{3.43}$$

Let $t \geq 1$ and r, γ be numbers such that

$$q \leq r \leq \infty, \quad 3\left(\frac{1}{q} - \frac{1}{r}\right) < 2, \quad 1 < \gamma < \min\left(q, \frac{3}{2}\right). \tag{3.44}$$

Then by the Sobolev embedding theorem, (3.38), (3.39) and (3.41) we have

$$\begin{aligned} \|I_2(t)\|_{L^r(\mathbb{R}^3)} &\leq C_{q,r} \int_{t-1}^t \|E(t-s) * \boldsymbol{K}(s)\|_{W^{2,q}(\mathbb{R}^3)}\, ds \\ &\quad + \int_0^{t-1} \|E(t-s) * \boldsymbol{K}(s)\|_{L^r(\mathbb{R}^3)}\, ds \\ &\leq C_{q,r} \int_{t-1}^t (t-s)^{-\frac{1}{2}} \|\boldsymbol{K}(s)\|_{W^{1,q}(\mathbb{R}^3)}\, ds \\ &\quad + C_{q,r} \int_0^{t-1} (t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{r}\right)} \|\boldsymbol{K}\|_{L^\gamma(\mathbb{R}^3)}\, ds \\ &\leq C_{q,r} \left\{ t^{-\frac{3}{2q}} + \int_0^{t-1} (t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{r}\right)} (1+s)^{-\frac{3}{2q}}\, ds \right\} \|\boldsymbol{f}\|_{L^q(\Omega)}. \end{aligned} \tag{3.45}$$

Observe that

$$\begin{aligned}\int_0^{t-1}(t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{r}\right)}(1+s)^{-\frac{3}{2q}}\,ds &\le C_{r,\gamma}\int_0^t(1+t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{r}\right)}(1+s)^{-\frac{3}{2q}}\,ds\\ &= C_{r,\gamma}\int_0^{t/2}(1+t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{r}\right)}(1+s)^{-\frac{3}{2q}}\,ds\\ &\quad + C_{r,\gamma}\int_0^{t/2}(1+\tau)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{r}\right)}(1+t-\tau)^{-\frac{3}{2q}}\,d\tau,\end{aligned}$$

where we have used the change of variable, $t-s=\tau$ in the second term in the last relation. When $0<s<t/2$, $1+t-s\ge 1+s$, we have

$$\begin{aligned}\int_0^{t-1}(t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{r}\right)}(1+s)^{-\frac{3}{2q}}\,ds &\le 2C_{r,\gamma}\left(1+\frac{t}{2}\right)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\int_0^{t/2}(1+s)^{-\frac{3}{2\gamma}}\,ds\\ &\le 2C_{q,r}(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\end{aligned}$$

because $3\gamma/2>1$ holds by (3.44), which combined with (3.45) implies that

$$\|I_2(t)\|_{L^r(\mathbb{R}^3)}\le C_{q,r}t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\boldsymbol{f}\|_{L^q(\Omega)}\quad\text{for any } t\ge 1 \tag{3.46}$$

provided that $q\le r\le\infty$ and $3(1/q-1/r)<2$. From (3.42) and (3.43) we have

$$\|\nabla I_1(t)\|_{L^q(\mathbb{R}^3)}\le C_q t^{-\frac{1}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)}. \tag{3.47}$$

By (3.38) and (3.41) we have

$$\begin{aligned}\|\nabla I_2(t)\|_{L^q(\mathbb{R}^3)} &\le C_q\int_{t-1}^t(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{3}{2q}}\,ds\|\boldsymbol{f}\|_{L^q(\Omega)}\\ &\quad + C_{q,r}\int_0^{t-1}(t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{q}\right)-\frac{1}{2}}(1+s)^{-\frac{3}{2q}}\,ds\|\boldsymbol{f}\|_{L^q(\Omega)}.\end{aligned} \tag{3.48}$$

Observe that

$$\begin{aligned}\int_0^{t-1}(t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{q}\right)-\frac{1}{2}}(1+s)^{-\frac{3}{2q}}\,ds &\le C_{q,\gamma}\int_0^t(1+t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{q}\right)-\frac{1}{2}}(1+s)^{-\frac{3}{2q}}\,ds\\ &= C_{q,\gamma}\int_0^{t/2}(1+t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{q}\right)-\frac{1}{2}}(1+s)^{-\frac{3}{2q}}\,ds\\ &\quad + C_{q,\gamma}\int_{t/2}^t(1+s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{q}\right)-\frac{1}{2}}(1+t-s)^{-\frac{3}{2q}}\,ds.\end{aligned}$$

If $1 < q \le 3$, then $3/2q - 1/2 \ge 0$, and therefore

$$\int_0^{t-1} (t-s)^{-\frac{3}{2}\left(\frac{1}{\gamma}-\frac{1}{q}\right)-\frac{1}{2}}(1+s)^{-\frac{3}{2q}}\,ds \le C_{q,\gamma}\left(1+\frac{t}{2}\right)^{-\frac{1}{2}}\int_0^{t/2}(1+s)^{-\frac{3}{2\gamma}}\,ds$$
$$\le C_{q,\gamma}(1+t)^{-\frac{1}{2}},$$

which combined with (3.48) implies that

$$\|\nabla I_2(t)\|_{L^q(\mathbb{R}^3)} \le C_q t^{-\frac{1}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)}, \quad t \ge 1 \tag{3.49}$$

provided that $1 < q \le 3$. The proof is completed. $\square$

3rd step

We consider the case when $0 < t \le 2$. We shall prove the following lemma.

Lemma 3.3. *Let $1 < q < \infty$ and $0 < t \le 2$ and $\boldsymbol{f} \in L^q_\sigma(\Omega)$. Then we have*

$$\|T(t)\boldsymbol{f}\|_{L^r(\Omega)} \le C_{q,r}t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\boldsymbol{f}\|_{L^q(\Omega)}, \qquad 1 < q \le r \le \infty, \tag{3.50}$$
$$\|\nabla T(t)\boldsymbol{f}\|_{L^r(\Omega)} \le C_{q,r}t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)}, \qquad 1 < q \le r < \infty. \tag{3.51}$$

Proof. For any real number $s \in (0,2)$, by complex interpolation theorem we have $W^{s,q}(\Omega) = [L^q(\Omega), W^{2,q}(\Omega)]_\theta$ with $s = 2\theta$ (see e.g., Triebel [22]). From (2.2) we have

$$\|T(t)\boldsymbol{f}\|_{L^q(\Omega)} \le C_q\|\boldsymbol{f}\|_{L^q(\Omega)}, \tag{3.52}$$
$$\|T(t)\boldsymbol{f}\|_{W^{2,q}(\Omega)} \le C_q t^{-1}\|\boldsymbol{f}\|_{L^q(\Omega)} \tag{3.53}$$

for $0 < t \le 2$. Therefore interpolating (3.52) and (3.53) for $s = 2\theta$ we obtain

$$\|T(t)\boldsymbol{f}\|_{W^{s,q}(\Omega)} \le C_{q,s}t^{-\frac{s}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)}. \tag{3.54}$$

From the Sobolev embedding theorem and (3.54), for $s = 3(1/q - 1/r)$ we have

$$\|T(t)\boldsymbol{f}\|_{L^r(\Omega)} \le C_{q,r}t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\boldsymbol{f}\|_{L^q(\Omega)} \tag{3.55}$$

for $0 < t \le 2$ and $1 < q \le r < \infty$. By (3.52) and (3.53) we have

$$\|\nabla T(t)\boldsymbol{f}\|_{L^q(\Omega)} \le C\|T(t)\boldsymbol{f}\|^{\frac{1}{2}}_{L^q(\Omega)}\|T(t)\boldsymbol{f}\|^{\frac{1}{2}}_{W^{2,q}(\Omega)} \le Ct^{-\frac{1}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)} \tag{3.56}$$

for $0 < t \le 2$. Therefore, by (3.55) and (3.56) we obtain

$$\|\nabla T(t)\boldsymbol{f}\|_{L^r(\Omega)} \le \left\|\nabla T\left(\frac{t}{2}+\frac{t}{2}\right)\boldsymbol{f}\right\|_{L^r(\Omega)} \le C\left(\frac{t}{2}\right)^{-\frac{1}{2}}\left\|T\left(\frac{t}{2}\right)\boldsymbol{f}\right\|_{L^r(\Omega)}$$
$$\le C_{q,r}t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}}\|\boldsymbol{f}\|_{L^q(\Omega)}$$

for $0 < t \leq 2$.

Finally we shall consider the L^∞ estimate. For $3 < q < \infty$, by using Sobolev's inequality:

$$\|\boldsymbol{u}\|_{L^\infty(\Omega)} \leq C\|\boldsymbol{u}\|^{\theta}_{W^{1,q}(\Omega)}\|\boldsymbol{u}\|^{1-\theta}_{L^q(\Omega)}$$

with $\theta = 3/q$ and (3.55) and (3.56) we have

$$\|T(t)\boldsymbol{f}\|_{L^\infty(\Omega)} \leq C_q t^{-\frac{3}{2q}}\|\boldsymbol{f}\|_{L^q(\Omega)} \tag{3.57}$$

for $0 < t \leq 2$. Next we consider the cases when $1 < q < 3/2$ or $3/2 < q < 3$. Let $3/(k+1) < q < 3/k$ with $k = 1, 2$. We set $\{q_\ell\}_{\ell=0}^k$ in such a way that $1/q_{\ell+1} = 1/q_\ell - 1/3$ $(\ell = 0, 1, \ldots, k-1)$ with $q_0 = q$. Since $1 < q < 3$, we see that $3 < q_k < \infty$. Therefore by using (3.57) with $q = q_k$ and (3.55) with $r = q_k$, we obtain

$$\begin{aligned}
\|T(t)\boldsymbol{f}\|_{L^\infty(\Omega)} &= \left\|T\left(\frac{t}{2}\right)T\left(\frac{t}{2}\right)\boldsymbol{f}\right\|_{L^\infty(\Omega)} \leq Ct^{-\frac{3}{2q_k}}\left\|T\left(\frac{t}{2}\right)\right\|_{L^{q_k}(\Omega)} \\
&\leq Ct^{-\frac{3}{2q_k}}t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{q_k}\right)}\|\boldsymbol{f}\|_{L^q(\Omega)} = Ct^{-\frac{3}{2q}}\|\boldsymbol{f}\|_{L^q(\Omega)},
\end{aligned}$$

for $t > 0$. This implies (3.57) for $1 - 3/q \notin \mathbb{N}_0$. When $1 - 3/q \in \mathbb{N}_0$, we choose r in such a way that $q < r < \infty$ and $1 - 3/r \notin \mathbb{N}_0$. Then, by (3.55) with $q = r$ and (3.57) we have

$$\begin{aligned}
\|T(t)\boldsymbol{f}\|_{L^\infty(\Omega)} &\leq C_r\left(\frac{t}{2}\right)^{-\frac{3}{2r}}\left\|T\left(\frac{t}{2}\right)\boldsymbol{f}\right\|_{L^r(\Omega)} \\
&\leq C_r\left(\frac{t}{2}\right)^{-\frac{3}{2r}}\left(\frac{t}{2}\right)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\boldsymbol{f}\|_{L^q(\Omega)} \leq C_{q,r}t^{-\frac{3}{2q}}\|\boldsymbol{f}\|_{L^q(\Omega)}
\end{aligned}$$

for $0 < t \leq 2$. Hence we get (3.55) for $1 < q \leq r \leq \infty$. The proof is completed. □

4th step

Now, we shall complete the proof of Theorem 1.1. Combining Lemma 3.2 and Lemma 3.3, we have

$$\|T(t)\boldsymbol{f}\|_{L^r(\Omega)} \leq C_{q,r}t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\boldsymbol{f}\|_{L^q(\Omega)} \tag{3.58}$$

for any $t > 0$ and $\boldsymbol{f} \in L^q_\sigma(\Omega)$ provided that $1 < q \leq r \leq \infty$ and $3(1/q - 1/r) < 2$. When $1 < q \leq r \leq \infty$ and $3(1/q - 1/r) \geq 2$, we choose numbers q_j, $j = 1, 2, \ldots, \ell-1$, in such a way that $q = q_0 < q_1 < q_2 < \cdots < q_{\ell-1} < q_\ell = r$ and $3(1/q_{m-1} - 1/q_m) < 2$

for $m = 1, 2, \dots, \ell$. Repeated use of (3.58) implies that

$$
\begin{aligned}
\|T(t)\boldsymbol{f}\|_{L^r(\Omega)} &= \Bigg\| T\Bigg(\underbrace{\frac{t}{\ell} + \cdots + \frac{t}{\ell}}_{\ell \text{ times}}\Bigg)\boldsymbol{f}\Bigg\|_{L^r(\Omega)} \\
&\le C_{q_\ell, q_{\ell-1}} \left(\frac{t}{\ell}\right)^{-\frac{3}{2}\left(\frac{1}{q_{\ell-1}} - \frac{1}{r}\right)} \Bigg\| T\Bigg(\underbrace{\frac{t}{\ell} + \cdots + \frac{t}{\ell}}_{\ell - 1 \text{ times}}\Bigg)\boldsymbol{f}\Bigg\|_{L^{q_{\ell-1}}(\Omega)} \\
&\le \cdots \le C_{q,r} \left(\frac{t}{\ell}\right)^{-\frac{3}{2}\left(\frac{1}{q} - \frac{1}{r}\right)} \|\boldsymbol{f}\|_{L^q(\Omega)},
\end{aligned}
$$

and therefore we have (3.58) for any $t > 0$ and $\boldsymbol{f} \in L^q_\sigma(\Omega)$ provided that $1 < q \le r \le \infty$.

Now we consider the case when $q = 1$. For any $\boldsymbol{\varphi}, \boldsymbol{\psi} \in C^\infty_{0,\sigma}(\Omega)$, by Theorem 2.5 and (3.58) we have

$$
|(T(t)\boldsymbol{\varphi}, \boldsymbol{\psi})_\Omega| = |(\boldsymbol{\varphi}, T(t)\boldsymbol{\psi})_\Omega| \le \|\boldsymbol{\varphi}\|_{L^1(\Omega)} \|T(t)\boldsymbol{\psi}\|_{L^\infty(\Omega)} \le C\|\boldsymbol{\varphi}\|_{L^1(\Omega)} t^{-3/2r'} \|\boldsymbol{\psi}\|_{L^{r'}(\Omega)},
$$

where $r' = r/(r-1)$, and therefore we have

$$
\|T(t)\boldsymbol{\varphi}\|_{L^r(\Omega)} \le C_r t^{-\frac{3}{2}\left(1 - \frac{1}{r}\right)} \|\boldsymbol{\varphi}\|_{L^1(\Omega)}. \tag{3.59}
$$

Since $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^{r'}_\sigma(\Omega)$, by the density argument we have (3.59) for any $\boldsymbol{\varphi} \in L^1_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^1(\Omega)}}$.

Combining (3.32) and (3.51), we obtain

$$
\|\nabla T(t)\boldsymbol{f}\|_{L^q(\Omega)} \le C_q t^{-\frac{1}{2}} \|\boldsymbol{f}\|_{L^q(\Omega)} \tag{3.60}
$$

for any $t > 0$ and $\boldsymbol{f} \in L^q_\sigma(\Omega)$ provided that $1 < q \le 3$. Combining (3.58), (3.59) and (3.60), we have

$$
\begin{aligned}
\|\nabla T(t)\boldsymbol{f}\|_{L^r(\Omega)} &= \left\| \nabla T\left(\frac{t}{2}\right) T\left(\frac{t}{2}\right) \boldsymbol{f} \right\|_{L^r(\Omega)} \le C_r t^{-\frac{1}{2}} \left\| T\left(\frac{t}{2}\right) \boldsymbol{f} \right\|_{L^r(\Omega)} \\
&\le C_{q,r} t^{-\frac{3}{2}\left(\frac{1}{q} - \frac{1}{r}\right) - \frac{1}{2}} \|\boldsymbol{f}\|_{L^q(\Omega)}.
\end{aligned}
$$

for any $t > 0$ and $\boldsymbol{f} \in L^q_\sigma(\Omega)$ provided that $1 \le q \le r \le 3$, $r \ne 1$. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. For notational simplicity, we use the abbreviation $\|\cdot\|_q$ which stands for $\|\cdot\|_{L^q(\Omega)}$. At first employing the argument due to Kato [11] for the Cauchy problem of the Navier-Stokes system, we shall solve the integral equations (INT) by contraction mapping principle.

In order to do this, we introduce the following symbols:

$$\begin{aligned}
[\boldsymbol{v}]_{\ell,q,t} &= \sup_{0<s\le t} s^{\ell}\|\boldsymbol{v}(s)\|_q, \\
[\![\boldsymbol{v}]\!]_t &= [\boldsymbol{v}]_{\frac{1-\delta}{2},\frac{3}{\delta},t} + [\nabla \boldsymbol{v}]_{\frac{1}{2},3,t}, \\
|\!|\!|\boldsymbol{v}|\!|\!|_t &= [\boldsymbol{v}]_{0,3,t} + [\boldsymbol{v}]_{\frac{1}{2},\infty,t} + [\![\boldsymbol{v}]\!]_t
\end{aligned}$$

with some fixed real number $\delta \in (0,1)$. As an underlying space, we set

$$\begin{aligned}
\mathcal{I}_M = \{(\boldsymbol{v}(t), \boldsymbol{B}(t)) \in BC([0,\infty); L^3_\sigma(\Omega) \times L^3_\sigma(\Omega)) \mid \\
\lim_{t\to 0+}\{[(\boldsymbol{v}-\boldsymbol{a}, \boldsymbol{B}-\boldsymbol{b})]_{0,3,t} + [(\boldsymbol{v},\boldsymbol{B})]_{\frac{1}{2},\infty,t} + [\![(\boldsymbol{v},\boldsymbol{B})]\!]_t\} = 0, \qquad (4.1)\\
\sup_{t>0} |\!|\!|(\boldsymbol{v},\boldsymbol{B})|\!|\!|_t \le 2M\|(\boldsymbol{a},\boldsymbol{b})\|_3\}, \qquad (4.2)
\end{aligned}$$

where M will be determined later (see (4.8) below). Set

$$\begin{aligned}
&\boldsymbol{v}_0(t) = e^{-tA}\boldsymbol{a}, \quad \boldsymbol{B}_0(t) = e^{-t\mathcal{M}}\boldsymbol{b}, \\
&\Phi(\boldsymbol{v},\boldsymbol{B})(t) = \begin{pmatrix} \boldsymbol{v}_0(t) \\ \boldsymbol{B}_0(t) \end{pmatrix} + \begin{pmatrix} F[\boldsymbol{v},\boldsymbol{B}](t) \\ G[\boldsymbol{v},\boldsymbol{B}](t) \end{pmatrix}.
\end{aligned}$$

We shall prove that there exist positive constants M and η such that if

$$\|(\boldsymbol{a},\boldsymbol{b})\|_3 \le \eta, \qquad (4.3)$$

then Φ becomes a contraction map from $\mathcal{I}_M$ into itself.

At the beginning, we shall show that

$$\lim_{t\to 0+}[(\boldsymbol{v}_0-\boldsymbol{a}, \boldsymbol{B}_0-\boldsymbol{b})]_{0,3,t} = 0, \qquad (4.4)$$

$$\lim_{t\to 0+}[\![(\boldsymbol{v}_0,\boldsymbol{B}_0)]\!]_t = 0, \qquad \lim_{t\to 0+}[(\boldsymbol{v}_0,\boldsymbol{B}_0)]_{\frac{1}{2},\infty,t} = 0. \qquad (4.5)$$

In fact, for any $\epsilon > 0$ there exists a pair $(\boldsymbol{a}_\epsilon, \boldsymbol{b}_\epsilon) \in C^\infty_{0,\sigma}(\Omega) \times C^\infty_{0,\sigma}(\Omega)$ so that $\|(\boldsymbol{a},\boldsymbol{b}) - (\boldsymbol{a}_\epsilon,\boldsymbol{b}_\epsilon)\|_3 < \epsilon$. Therefore, by the L^3-boundedness of the semigroups (Theorems 1.1

and 1.3 with $q = r = 3$), we see that

$$
\begin{aligned}
\|(\boldsymbol{v}_0(t), \boldsymbol{B}_0(t)) - (\boldsymbol{a}, \boldsymbol{b})\|_3 &\le \|(e^{-tA}(\boldsymbol{a} - \boldsymbol{a}_\epsilon), e^{-t\mathcal{M}}(\boldsymbol{b} - \boldsymbol{b}_\epsilon))\|_3 \\
&\quad + \|(e^{-tA}\boldsymbol{a}_\epsilon - \boldsymbol{a}_\epsilon, e^{-t\mathcal{M}}\boldsymbol{b}_\epsilon - \boldsymbol{b})\|_3 + \|(\boldsymbol{a}_\epsilon - \boldsymbol{a}, \boldsymbol{b}_\epsilon - \boldsymbol{b})\|_3 \\
&\le C\epsilon + \|(e^{-tA}\boldsymbol{a}_\epsilon - \boldsymbol{a}_\epsilon, e^{-t\mathcal{M}}\boldsymbol{b}_\epsilon - \boldsymbol{b})\|_3 \\
&\le C\epsilon + \int_0^t \left\| \frac{d}{ds}(e^{-sA}\boldsymbol{a}_\epsilon, e^{-s\mathcal{M}}\boldsymbol{b}_\epsilon) \right\|_3 ds \\
&\le C\epsilon + Ct\|(\boldsymbol{a}_\epsilon, \boldsymbol{b}_\epsilon)\|_{W^{2,3}(\Omega)}.
\end{aligned}
$$

Therefore we have

$$
\lim_{t \to 0+} [(\boldsymbol{v}_0 - \boldsymbol{a}, \boldsymbol{B}_0 - \boldsymbol{b})]_{0,3,t} \le C\epsilon.
$$

This implies (4.4), because ϵ is chosen arbitrarily. By similar manner, we have

$$
\begin{aligned}
t^{\frac{1-\delta}{2}} \|(\boldsymbol{v}_0(t), \boldsymbol{B}_0(t))\|_{\frac{3}{\delta}} &\le t^{\frac{1-\delta}{2}} \|(e^{-tA}(\boldsymbol{a} - \boldsymbol{a}_\epsilon), e^{-t\mathcal{M}}(\boldsymbol{b} - \boldsymbol{b}_\epsilon))\|_{\frac{3}{\delta}} \\
&\quad + t^{\frac{1-\delta}{2}} \|(e^{-tA}\boldsymbol{a}_\epsilon, e^{-t\mathcal{M}}\boldsymbol{b}_\epsilon)\|_{\frac{3}{\delta}} \\
&\le C\|(\boldsymbol{a} - \boldsymbol{a}_\epsilon, \boldsymbol{b} - \boldsymbol{b}_\epsilon)\|_3 + Ct^{\frac{1}{2} - \frac{3}{2r}} \|(\boldsymbol{a}_\epsilon, \boldsymbol{b}_\epsilon)\|_r \\
&\le C\epsilon + Ct^{\frac{1}{2} - \frac{3}{2r}} \|(\boldsymbol{a}_\epsilon, \boldsymbol{b}_\epsilon)\|_r
\end{aligned}
$$

with some $r \in (3, 3/\delta)$, which implies

$$
\lim_{t \to 0+} [\boldsymbol{v}_0, \boldsymbol{B}_0]_{\frac{1-\delta}{2}, \frac{3}{\delta}, t} \le C\epsilon. \tag{4.6}
$$

From similar calculation, we see that

$$
\lim_{t \to 0+} [(\boldsymbol{v}_0, \boldsymbol{B}_0)]_{\frac{1}{2}, \infty, t} \le C\epsilon, \qquad \lim_{t \to 0+} [\nabla(\boldsymbol{v}_0, \boldsymbol{B}_0)]_{\frac{1}{2}, 3, t} \le C\epsilon. \tag{4.7}
$$

Since ϵ is chosen arbitrarily, by (4.6) and (4.7) we have (4.5).

By Theorems 1.1 and 1.3, one can easily see that

$$
|\!|\!|(\boldsymbol{v}_0, \boldsymbol{B}_0)|\!|\!|_t \le M\|(\boldsymbol{a}, \boldsymbol{b})\|_3 \quad \text{for any } t > 0 \tag{4.8}
$$

with some constant M. In particular, from (4.6), (4.7) and (4.8), we see that $(\boldsymbol{v}_0(t), \boldsymbol{B}_0(t)) \in \mathcal{I}_M$.

Now, we shall estimate the nonlinear terms $F[\boldsymbol{v}, \boldsymbol{B}](t)$ and $G[\boldsymbol{v}, \boldsymbol{B}](t)$. In order to do this, we prepare the following inequality essentially due to the Hölder inequality:

$$
\|(\boldsymbol{u}(s) \cdot \nabla)\boldsymbol{v}(s)\|_{\frac{3}{1+\delta}} \le \|\boldsymbol{u}(s)\|_{\frac{3}{\delta}} \|\nabla \boldsymbol{v}(s)\|_3 \le Cs^{-1+\frac{\delta}{2}} [\![\boldsymbol{u}]\!]_t [\![\boldsymbol{v}]\!]_t \tag{4.9}
$$

for any $0 < s \leq t$. By Theorem 1.3 and the L^q-boundedness of the Helmholtz projection (1.3), we have

$$\begin{aligned}
\|F[\boldsymbol{v},\boldsymbol{B}](t)\|_3 &\leq \int_0^t \|e^{-(t-s)A}P[(\boldsymbol{v}(s)\cdot\nabla)\boldsymbol{v}(s)-(\boldsymbol{B}(s)\cdot\nabla)\boldsymbol{B}(s)]\|_3\,ds \\
&\leq C\int_0^t (t-s)^{-\frac{3}{2}\left(\frac{1+\delta}{3}-\frac{1}{3}\right)}(\|(\boldsymbol{v}(s)\cdot\nabla)\boldsymbol{v}(s)\|_{\frac{3}{1+\delta}}+\|(\boldsymbol{B}(s)\cdot\nabla)\boldsymbol{B}(s)\|_{\frac{3}{1+\delta}})\,ds \\
&\leq C\int_0^t (t-s)^{-\frac{\delta}{2}}(\|\boldsymbol{v}(s)\|_{\frac{3}{\delta}}\|\nabla\boldsymbol{v}(s)\|_3+\|\boldsymbol{B}(s)\|_{\frac{3}{\delta}}\|\nabla\boldsymbol{B}(s)\|_3)\,ds.
\end{aligned}$$

By similar manner with Theorem 1.1, we have

$$\|G[\boldsymbol{v},\boldsymbol{B}](t)\|_3 \leq C\int_0^t (t-s)^{-\frac{\delta}{2}}(\|\boldsymbol{v}(s)\|_{\frac{3}{\delta}}\|\nabla\boldsymbol{B}(s)\|_3+\|\boldsymbol{B}(s)\|_{\frac{3}{\delta}}\|\nabla\boldsymbol{v}(s)\|_3)\,ds.$$

From the above two estimates and (4.9), we obtain

$$\begin{aligned}
\|(F[\boldsymbol{v},\boldsymbol{B}])(t),G[\boldsymbol{v},\boldsymbol{B}](t)\|_3 &\leq C\int_0^t (t-s)^{-\frac{\delta}{2}}\|(\boldsymbol{v}(s),\boldsymbol{B}(s))\|_{\frac{3}{\delta}}\|\nabla(\boldsymbol{v}(s),\boldsymbol{B}(s))\|_3\,ds \\
&\leq C\int_0^t (t-s)^{-\frac{\delta}{2}}s^{-1+\frac{\delta}{2}}\,ds[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2 \\
&= CB\left(1-\frac{\delta}{2},\frac{\delta}{2}\right)[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2,
\end{aligned} \tag{4.10}$$

where $B(q,r)$ denotes the beta function. From similar calculations, we obtain the following estimates:

$$\begin{aligned}
\|(F[\boldsymbol{v},\boldsymbol{B}](t),G[\boldsymbol{v},\boldsymbol{B}](t)\|_{\frac{3}{\delta}} &\leq C\int_0^t (t-s)^{-\frac{1}{2}}s^{-1+\frac{\delta}{2}}\,ds[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2 \\
&\leq CB\left(\frac{1}{2},\frac{\delta}{2}\right)t^{-\frac{1-\delta}{2}}[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2; && (4.11)\\
\|\nabla(F[\boldsymbol{v},\boldsymbol{B}](t),G[\boldsymbol{v},\boldsymbol{B}](t)\|_3 &\leq C\int_0^t (t-s)^{-\frac{1+\delta}{2}}s^{-1+\frac{\delta}{2}}\,ds[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2 \\
&\leq CB\left(\frac{1-\delta}{2},\frac{\delta}{2}\right)t^{-\frac{1}{2}}[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2; && (4.12)\\
\|(F[\boldsymbol{v},\boldsymbol{B}](t),G[\boldsymbol{v},\boldsymbol{B}](t)\|_\infty &\leq C\int_0^t (t-s)^{-\frac{1+\delta}{2}}s^{-1+\frac{\delta}{2}}\,ds[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2 \\
&\leq CB\left(\frac{1-\delta}{2},\frac{\delta}{2}\right)t^{-\frac{1}{2}}[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2. && (4.13)
\end{aligned}$$

From (4.10), (4.11), (4.12) and (4.13), we have

$$\|\!|(F[\boldsymbol{v},\boldsymbol{B}],G[\boldsymbol{v},\boldsymbol{B}])\|\!|_t \le C[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2. \tag{4.14}$$

Hence, from (4.8) and (4.14), we have

$$\|\!|\Phi(\boldsymbol{v},\boldsymbol{B})\|\!|_t \le M\|(\boldsymbol{a},\boldsymbol{b})\|_3 + C[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2, \tag{4.15}$$

$$\begin{aligned}&[\Phi(\boldsymbol{v},\boldsymbol{B})-(\boldsymbol{a},\boldsymbol{b})]_{0,3,t} + [\Phi(\boldsymbol{v},\boldsymbol{B})]_{\frac{1}{2},\infty,t} + [\![\Phi(\boldsymbol{v},\boldsymbol{B})]\!]_t \\ &\le [(\boldsymbol{v}_0-\boldsymbol{a},\boldsymbol{B}_0-\boldsymbol{b})]_{0,3,t} + [(\boldsymbol{v}_0,\boldsymbol{B}_0)]_{\frac{1}{2},\infty,t} + [\![(\boldsymbol{v}_0,\boldsymbol{B}_0)]\!]_t + C[\![(\boldsymbol{v},\boldsymbol{B})]\!]_t^2.\end{aligned} \tag{4.16}$$

Therefore, if $(\boldsymbol{v},\boldsymbol{B}) \in \mathcal{I}_M$, then by (4.1), (4.2), (4.4), (4.5), (4.15) and (4.16), we obtain

$$\|\!|\Phi(\boldsymbol{v},\boldsymbol{B})\|\!|_t \le M\|(\boldsymbol{a},\boldsymbol{b})\|_3 + 4CM^2\|(\boldsymbol{a},\boldsymbol{b})\|_3^2 \quad \text{for any } t>0, \tag{4.17}$$

$$\lim_{t\to 0+}([\Phi(\boldsymbol{v},\boldsymbol{B})-(\boldsymbol{a},\boldsymbol{b})]_{0,3,t} + [\Phi(\boldsymbol{v},\boldsymbol{B})]_{\frac{1}{2},\infty,t} + [\![\Phi(\boldsymbol{v},\boldsymbol{B})]\!]_t) = 0. \tag{4.18}$$

Choose an $\eta > 0$ in such a way that

$$4CM\eta < 1. \tag{4.19}$$

Then by (4.17) we have

$$\|\!|\Phi(\boldsymbol{v},\boldsymbol{B})\|\!|_t < 2M\|(\boldsymbol{a},\boldsymbol{b})\|_3 \quad \text{for any } t>0 \tag{4.20}$$

provided that $\|(\boldsymbol{a},\boldsymbol{b})\|_3 \le \eta$, which combined with (4.18) implies that $\Phi(\boldsymbol{v},\boldsymbol{B}) \in \mathcal{I}_M$ provided that $(\boldsymbol{v},\boldsymbol{B}) \in \mathcal{I}_M$. This shows that Φ is a mapping from $\mathcal{I}_M$ into itself. By using (4.9), Theorems 1.1 and 1.3 and employing the same argument as in the proof of (4.14), we have

$$\begin{aligned}&\|\!|\Phi(\boldsymbol{v}_1,\boldsymbol{B}_1)-\Phi(\boldsymbol{v}_2,\boldsymbol{B}_2)\|\!|_t \\ \le\,& C([\![(\boldsymbol{v}_1,\boldsymbol{B}_1)]\!]_t + [\![(\boldsymbol{v}_2,\boldsymbol{B}_2)]\!]_t)[\![(\boldsymbol{v}_1,\boldsymbol{B}_1)-(\boldsymbol{v}_2,\boldsymbol{B}_2)]\!]_t \\ \le\,& 4CM\|(\boldsymbol{a},\boldsymbol{b})\|_3\|\!|(\boldsymbol{v}_1,\boldsymbol{B}_1)-(\boldsymbol{v}_2,\boldsymbol{B}_2)\|\!|_t\end{aligned} \tag{4.21}$$

for any $(\boldsymbol{v}_1,\boldsymbol{B}_1),(\boldsymbol{v}_2,\boldsymbol{B}_2) \in \mathcal{I}_M$. If we choose an $\eta > 0$ in such a way that

$$4CM\eta < \frac{1}{2},$$

then it follows from (4.21) that Φ is a contraction map from $\mathcal{I}_M$ into itself if $\|(\boldsymbol{a},\boldsymbol{b})\|_3 \le \eta$. Therefore, there exists a unique fixed point $(\boldsymbol{v}(t),\boldsymbol{B}(t)) \in \mathcal{I}_M$ of Φ, which solves (INT). The uniqueness of solutions to (INT) holds for any $(\boldsymbol{v}(t),\boldsymbol{B}(t)) \in \mathcal{I}_M$. Namely, if $(\boldsymbol{v}_1(t),\boldsymbol{B}_1(t)),(\boldsymbol{v}_2(t),\boldsymbol{B}_2(t)) \in \mathcal{I}_M$ satisfy the integral equations (INT)

with the same initial data $(\boldsymbol{a}, \boldsymbol{b}) \in L^3_\sigma(\Omega) \times L^3_\sigma(\Omega)$ with $\|(\boldsymbol{a}, \boldsymbol{b})\|_3 \leq \eta$, then we have $(\boldsymbol{v}_1(t), \boldsymbol{B}_1(t)) = (\boldsymbol{v}_2(t), \boldsymbol{B}_2(t))$ for any $t > 0$.

Now we shall show sharp asymptotic behavior of the global in time strong solution: (1.4) and (1.5). In order to do this, at first we shall show the following:

$$\lim_{t\to\infty} \|(\boldsymbol{v}(t), \boldsymbol{B}(t))\|_3 = 0. \tag{4.22}$$

Given $0 < \gamma < 1/2$, we take $3/2 < q < 3$ such that $\gamma = 3/2q - 1/2$. Given $(\boldsymbol{a}, \boldsymbol{b}) \in C^\infty_{0,\sigma}(\Omega) \times C^\infty_{0,\sigma}(\Omega)$ with $\|(\boldsymbol{a}, \boldsymbol{b})\|_3 < \eta$, let $(\boldsymbol{v}(t), \boldsymbol{B}(t))$ be solution of (INT). Then applying the L^q-L^3 estimate and the $L^{3/2}$-L^3 estimate for e^{-tA} and $e^{-t\mathcal{M}}$ to (INT) and using the Hölder inequality, we have

$$\begin{aligned}
&\|(\boldsymbol{v}(t), \boldsymbol{B}(t))\|_3 \\
&\leq Ct^{-\gamma}\|(\boldsymbol{a}, \boldsymbol{b})\|_q + C\int_0^t (t-s)^{-\frac{1}{2}}\|(\boldsymbol{v}(s), \boldsymbol{B}(s))\|_3\|\nabla(\boldsymbol{v}(s), \boldsymbol{B}(s))\|_3\, ds \\
&\leq Ct^{-\gamma}\|(\boldsymbol{a}, \boldsymbol{b})\|_q + C\int_0^t (t-s)^{-\frac{1}{2}} s^{-\gamma} s^{-\frac{1}{2}}\, ds [(\boldsymbol{v}, \boldsymbol{B})]_{\gamma,3,t}[\nabla(\boldsymbol{v}, \boldsymbol{B})]_{\frac{1}{2},3,t} \\
&\leq Ct^{-\gamma}\left\{\|(\boldsymbol{a}, \boldsymbol{b})\|_q + CB\left(\frac{1}{2}, \frac{1}{2} - \gamma\right)\|(\boldsymbol{a}, \boldsymbol{b})\|_3\|[(\boldsymbol{v}, \boldsymbol{B})]_{\gamma,3,t}\right\},
\end{aligned}$$

which implies that

$$[(\boldsymbol{v}, \boldsymbol{B})]_{\gamma,3,t} \leq C\|(\boldsymbol{a}, \boldsymbol{b})\|_q + C\|(\boldsymbol{a}, \boldsymbol{b})\|_3[(\boldsymbol{v}, \boldsymbol{B})]_{\gamma,3,t}.$$

Since choosing $\eta > 0$ smaller if necessary, we may assume that $C\|(\boldsymbol{a}, \boldsymbol{b})\|_3 \leq 1/2$ provided that $\|(\boldsymbol{a}, \boldsymbol{b})\|_3 \leq \eta$, we have

$$[(\boldsymbol{v}, \boldsymbol{B})]_{\gamma,3,t} \leq 2C\|(\boldsymbol{a}, \boldsymbol{b})\|_q.$$

This implies that (4.22) holds for any initial data $(\boldsymbol{a}, \boldsymbol{b}) \in C^\infty_{0,\sigma}(\Omega) \times C^\infty_{0,\sigma}(\Omega)$ with $\|(\boldsymbol{a}, \boldsymbol{b})\|_3 \leq \eta$.

For general $(\boldsymbol{a}, \boldsymbol{b}) \in L^3_\sigma(\Omega) \times L^3_\sigma(\Omega)$ with $\|(\boldsymbol{a}, \boldsymbol{b})\|_3 < \eta$ and any $\epsilon > 0$, we choose $\boldsymbol{a}_\epsilon$ and $\boldsymbol{b}_\epsilon$ in such a way that $\|(\boldsymbol{a}_\epsilon - \boldsymbol{a}, \boldsymbol{b}_\epsilon - \boldsymbol{b})\|_3 \leq \epsilon$. Choosing $\epsilon > 0$ smaller if necessary, we may assume that $\|(\boldsymbol{a}_\epsilon, \boldsymbol{b}_\epsilon)\|_3 < \eta$ for any $\epsilon > 0$. Since $\|(\boldsymbol{a}_\epsilon, \boldsymbol{b}_\epsilon)\|_3 < \eta$, the corresponding solution of (INT) satisfies (4.22). Combining this fact and continuous dependence of solution: $L^3_\sigma(\Omega) \times L^3_\sigma(\Omega) \ni (\boldsymbol{a}, \boldsymbol{b}) \mapsto (\boldsymbol{v}(t), \boldsymbol{B}(t)) \in BC([0,\infty); L^3_\sigma(\Omega) \times L^3_\sigma(\Omega))$, we have

$$\begin{aligned}
\|(\boldsymbol{v}(t), \boldsymbol{B}(t))\|_3 &\leq \|(\boldsymbol{v}(t) - \boldsymbol{v}_\epsilon(t), \boldsymbol{B}(t) - \boldsymbol{B}_\epsilon(t))\|_3 + \|(\boldsymbol{v}_\epsilon(t), \boldsymbol{B}_\epsilon(t))\|_3 \\
&\leq C\epsilon + C\|(\boldsymbol{v}_\epsilon(t), \boldsymbol{B}_\epsilon(t))\|_3.
\end{aligned}$$

Since ϵ is arbitrary and $(\boldsymbol{v}_\epsilon(t), \boldsymbol{B}_\epsilon(t))$ satisfies (4.22), we get (4.22) for any initial data $(\boldsymbol{a}, \boldsymbol{b}) \in L^3_\sigma(\Omega) \times L^3_\sigma(\Omega)$ with $\|(\boldsymbol{a}, \boldsymbol{b})\|_3 \leq \eta$.

By the interpolation inequality, we get

$$
\begin{aligned}
t^{\frac{1}{2}-\frac{3}{2q}}\|(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_q &\le \|(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_3^{\theta}\left(t^{\frac{1}{2}}\|(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_\infty\right)^{1-\theta}\\
&\le C_q\|(\boldsymbol{a},\boldsymbol{b})\|_3^{1-\theta}\|(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_3^{\theta}
\end{aligned}
$$

with $1/q=\theta/3$, which together with (4.22) implies that (1.4) for $3<q<\infty$. Here we have used the global boundedness of $t^{1/2}\|(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_\infty$ which is guaranteed by the fact that a pair $(\boldsymbol{v}(t),\boldsymbol{B}(t))$ is global solution of (INT) with property (4.1) and (4.2). Finally, we shall prove (1.4) for $q=\infty$ and (1.5). In order to do this, we rewrite (INT) as follows:

$$
\begin{cases}
\boldsymbol{v}(t)=e^{-\frac{t}{2}A}\boldsymbol{v}(t/2)-\displaystyle\int_{\frac{t}{2}}^{t}e^{-(t-s)A}P[(\boldsymbol{v}(s)\cdot\nabla)\boldsymbol{v}(s)-(\boldsymbol{B}(s)\cdot\nabla)\boldsymbol{B}(s)]\,ds,\\
\boldsymbol{B}(t)=e^{-\frac{t}{2}\mathcal{M}}\boldsymbol{B}(t/2)-\displaystyle\int_{\frac{t}{2}}^{t}e^{-(t-s)\mathcal{M}}[(\boldsymbol{v}(s)\cdot\nabla)\boldsymbol{B}(s)-(\boldsymbol{B}(s)\cdot\nabla)\boldsymbol{v}(s)]\,ds.
\end{cases}
$$

Then by Theorems 1.1 and 1.3 we obtain

$$
\begin{aligned}
\|(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_\infty \le\, &Ct^{-\frac{1}{2}}\,\|(\boldsymbol{v}(t/2),\boldsymbol{B}(t/2))\|_3\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4}}\|(\boldsymbol{v}(s),\boldsymbol{B}(s))\|_6\|\nabla(\boldsymbol{v}(s),\boldsymbol{B}(s))\|_3\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_3 \le\, &Ct^{-\frac{1}{2}}\,\|(\boldsymbol{v}(t/2),\boldsymbol{B}(t/2))\|_3\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4}}\|(\boldsymbol{v}(s),\boldsymbol{B}(s))\|_6\|\nabla(\boldsymbol{v}(s),\boldsymbol{B}(s))\|_3\,ds
\end{aligned}
$$

Therefore combining the above two estimates and $\|\nabla(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_3\le Ct^{-1/2}\|(\boldsymbol{a},\boldsymbol{b})\|_3$, we obtain

$$
\begin{aligned}
&t^{\frac{1}{2}}(\|(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_\infty+\|\nabla(\boldsymbol{v}(t),\boldsymbol{B}(t))\|_3)\\
&\quad\le C\|(\boldsymbol{v}(t/2),\boldsymbol{B}(t/2))\|_3+C\|(\boldsymbol{a},\boldsymbol{b})\|_3\sup_{t/2\le s\le t}s^{\frac{1}{4}}\|(\boldsymbol{v}(s),\boldsymbol{B}(s))\|_6
\end{aligned}
$$

for $t>0$. Therefore, from (4.22) and (1.4) with $q=6$, we have (1.4) for $q=\infty$ and (1.5). This completes the proof of Theorem 1.4.

Acknowledgments. The author would like to express his heartily gratitude to Professor Yoshihiro Shibata for valuable comments and constant encouragements. The author is also grateful to the referee for many valuable comments and helpful suggestions.

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